18.152 PROBLEM SET 2

due February 28th 9:30 am (Gradescope).

You can collaborate with other students when working on problems. However, you should write the solutions using your own words and thought.

Problem 1. *Let $u(x,t)$ be a solution to the heat equation on $\overline{Q_t} = \{(x,t) : 0 \le x \le L, 0 \le t \le T\}$ satisfying $-u_x(0,t) = u_x(L,t) \le 0$ [resp. ≥ 0] and $u(x,0) = g(x)$. Prove*

$$\max_{\overline{Q_T}} u(x,t) \le \max_{0\le x\le L} g(x) \qquad [\textit{resp.} \min_{\overline{Q_T}} u(x,t) \ge \min_{0\le x\le L} g(x)].$$

Hint: Show that $u(0,t), u(L,t) \le \max g$ (for the first case) by modifying the proof of the maximum principle.

Problem 2. *Let $u(x,t)$ satisfy $u_t(x,t) = u_{xx}(x,t) + f(x,t)$ and $-u_x(0,t) = u_x(L,t) = 0$.*

(1) *Show that* $|u(x,t)| \le \max_{0\le x\le L} |g(x)| + t \max_{(x,s)\in\overline{Q_t}} |f(x,s)|$.

(2) *Show that* $w = \frac{t}{t+1}|u_x|^2 + \frac{1}{2}u^2 - At$ *is a subsolution to the heat equation in* $\overline{Q_T}$, *where* $A = \max_{\overline{Q_T}} \big(|f_x|^2 + |f||u|\big)$.

(3) (*Bonus problem*) *Establish an upper bound for* $|u_x(x,t)|^2$ *where* $t > 0$ *in terms of* f_x, f, g, *and* t.

Hint(1): Show that $v(x,s) = u(x,s) - As$ is a subsolution to the heat equation, where $A = \max_{\overline{Q_t}} |f|$. Then, apply the result in Problem 1.

Problem 3. *Let $u(x,y,t)$ be a solution to the heat equation. Check that $w = \frac{t}{t+1}|\nabla^2 u|^2 + \frac{1}{2}|\nabla u|^2$ is a subsolution, where $|\nabla^2 u|^2 = u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2$.*

Problem 4. *Let $u(x,t)$ be a solution to the heat equation on $\mathbb{R} \times [0,T]$ satisfying $|u(x,t)| \le C$ on $\mathbb{R} \times [0,T]$ for some constant C and also satisfying $\lim_{|x|\to+\infty} |u_x(x,0)| = 0$. Prove that $\lim_{|x|\to+\infty} |u_x(x,t)| = 0$ holds for all $t \in [0,T]$.*

Hint(1): Apply the maximum principle to $w = \eta^2 |u_x|^2 + 20R^2u^2$ where

$$\eta = (R^2 - |x-2R|^2)_+ = \max\{0, R^2 - |x-2R|^2\}.$$

Then, pass R to $+\infty$ and $-\infty$.

Problem 5 (Bonus)**.** *Let $u(x,y,t)$ be a solution to the heat equation on $\Omega \times [0,T]$ satisfying $u = 0$ on $\partial\Omega \times [0,T]$ and $u(x,y,0) = g(x,y)$. Suppose that $\partial\Omega$ is a smooth curve. Establish an upper bound for the gradient $|\nabla u|$ in terms of g, ∇g, and the minimum of radius of circles contacting $\partial\Omega$ from outside.*

Consider a circle outside Ω touching a point (x_0,y_0) of $\partial\Omega$. Put the center of the circle at the origin by translation. And construct a rotationally symmetric function $\varphi(x,y)$ satisfying $\varphi(x_0,y_0) = 0$, $\Delta\varphi \le 0$, and $\varphi \ge g$.